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Numerical Experiments on Compact Computational Schemes for Solving the First Biharmonic Problem in Rectangles

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DEDICATED TO THE MEMORY OF MY FATHER THEODOR

Compact computational schemes for the first biharmonic problem in a rectangle $a \times b$ with fourth- and second-order truncation errors and expressed in matrix form are presented. The matrix formulation of the fourth-order schemes is based on a kind of (p, q, r, s) non-coupled approach which may be viewed as an extension of the known non-coupled (p, q) method—see [6]—that uses 9-point stencils to approximate the Laplacian at the $l \times m$ interior nodes of a grid, with l and m standing for the number of equidistant subdivision points of the edges a and b, respectively. The final matrix equation which serves as a fourth-order discrete equivalent of the problem may be expressed in conventional formulation as a symmetric system with $l \times m$ unknowns. For second-order schemes, the matrix equation in its initial form is based on the non-coupled (p, q) approach employing 5-point stencils to approximate the Laplacian at the $l \times m$ interior grid nodes, while the final system in its conventional formulation is again symmetric. For the specific values of p, q, r, and s used in this paper, namely $p = 1$, $q = 2$, $r = 3$, and $s = 4$, it is possible to solve the problem by means of a quasi direct method after reducing the solution of both fourth- and second-order schemes to the solution of two symmetric and positive definite linear systems each of order equal to $\min(l, m)$. In addition, for the same values of p, q, r, and s, the employment of the SOR iterarive method leads, after a reasonable number of iterations, to results which agree with the ones obtained by the already mentioned quasi-direct method of solution. The experimentally observed accuracy for schemes with fourth-order truncation error (at least for the problems considered in this paper) was also of fourth order. For schemes with second-order truncation errors the observed accuracy was also of second order, as it should be, since the schemes in question express in effect the (p,q) approach for which—see [5, 7]—a formal proof of accuracy exists. © 1994 Academic Press, Inc.

1. INTRODUCTION

In solving numerically the first problem of the biharmonic equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)^2 \cdot \phi = f(x, y) \tag{1}$$

in a rectangle $a \times b$, where both ϕ and its normal derivative $\partial\phi/\partial n$ take prescribed values on the boundary, one is faced with the fact that the system of discrete equations approximating (1) is ill-conditioned.

There are both iterative [5] and direct [6] second-order methods for solving the above system of discrete equations which use 5-point stencils to approximate the Laplace operator and are based on the so-called (p, q) approach. In [5] the discrete model of the problem is constructed by combining the discrete models—coupled (p, q) approach—of the two Dirichlet problems,

$$\nabla^2 g = f \tag{2a}$$

$$\nabla^2 \phi = g, \tag{2b}$$

and is solved by means of the SOR iterative method.

On the other hand, such a coupling is avoided in [6] by incorporating the formulas for expressing the boundary conditions in a unified discrete model—non-coupled (p, q) approach—which is solved directly. Although the direct method of solution by employing fairly reliable band solvers is more accurate and more efficient than the iterative one, it is stated in [6] that there is a problem of storage requirements concerning the coefficient matrix of the system to be solved. Actually, due to storage limitations on an IBM 370/158 mainframe computer, no mesh sizes of h lower than $\frac{1}{20}$ were used.

In addition to the above second-order methods there is a method utilizing 9- or 5-point stencils which is due to Stephenson [10] and is characterized by fourth- or second-order accuracy, respectively. As far as the author knows the latest method which is based on the Hermite approach (Mehrstellenverfahren) gives the best results in solving the biharmonic problem in rectangles.

The method in question uses three nodal parameters (ϕ, $\partial\phi/\partial x$, $\partial\phi/\partial y$) at each of the $l \times m$ interior nodes of a rectangular grid and gives excellent results when the $3l \times 3m$ equations are solved directly. However, when iterative procedures were used, convergence was very slow and there was also a case where SOR failed to converge after 10,000 iterations. While the direct solution of the $3l \times 3m$ equations shows the above-

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mentioned excellent results, there is always the problem of memory requirements for storing the elements of the band matrix even for not very large values of l and m (e.g., for l and m greater than 16), since each node is now associated with a triplet of unknown parameters. Another drawback of the method which is in fact stated in [10] is that "solution to the resulting linear system of equations cannot be obtained quickly because of lack of simple structure and positive definitness."

The schemes presented in this paper, which again may be of fourth- or second-order, are based on a kind of non-coupled (p, q, r, s) approach and aim at avoiding, as far as possible, the weak points of numerical processing reported in [6, 10] by giving the most possible accurate results at a moderate computing cost. More precisely the solution of the final transformed version of the system of $l \times m$ equations, which turns out to be symmetric and—at least for the special 4-tuple (1, 2, 3, 4) of the computational indices p, q, r, and s—with positive diagonal entries as well, may be reduced to the solution of two symmetric and positive definite systems of order equal to $\min(l, m)$. Under the assumption that the coefficient matrix of the initial system of the $l \times m$ equations is positive definite, one may conclude by means of the Ostrowski–Reich theorem [2] that the employment of the SOR iterative procedure for solving the above transformed version of it, will result in convergence.

The latest assumption seems more or less reasonable since intuition suggests that each of the eigenvalues of the initial discrete model of the problem (which obviously coincides with an eigenvalue of its transformed version) is in effect an approximation to a positive multiple of one of the natural frequencies of an $a \times b$ rectangular plate with clamped edges. Anyhow, irrespectively of the validity of this assumption, the numerical experiments showed convergence after a reasonable number of iterations (as a matter of fact, considerably less than those cited in [5]) in all cases where the SOR procedure was applied.

However, at this point one should note that the results obtained by the methods introduced in this paper, while confirming those cited in [5, 6] and without being unacceptable, are roughly one order of magnitude less accurate than the corresponding ones recorded in [10]. According to the author's opinion this is mainly due to the fact that Stephenson's model with its three nodal unknown parameters constitutes a much better substitute for the original problem than models with only one nodal parameter. An additional but less significant reason might be the use of a DEC-20 computer with accuracy between 10^{-7} and 10^{-8}—see [2]—in conducting the experiments in [10] in contrast to the XT personal computer (an AMSTRAD 1512) with accuracy between 10^{-6} and 10^{-7} which was used to derive the results reported in this paper.

In presenting compact schemes for the first biharmonic problem, the material is organized as follows:

The structure of the schemes with fourth-order truncation errors are described first, since those with second-order errors may be viewed as a special case of the former ones. More precisely, the basic matrix equation of the fourth-order model with the $l \times m$ matrix of ϕ-values at the grid nodes as unknown, is initially introduced using the already mentioned non-coupled (p, q, r, s) approach to approximate normal derivatives at the boundaries. Next, a series of transformations is applied to this equation, resulting in an equivalent symmetric final linear model. As it has been already stated, for $(p, q, r, s) = (1, 2, 3, 4)$ the diagonal entries of this symmetric system are positive while the solution of the system itself is ultimately reduced to the solution of two symmetric and positive definite systems of order equal to $\min(l, m)$. A description of such a reduction, which may be regarded as a quasi-direct method of solution of the original problem, follows, while the final part of the material concerning the fourth-order discretization schemes consists of a series of numerical experiments. These experiments which revealed fourth-order accuracy, are related either to procedures serving to test the reliability of the above quasi-direct method or to gain experience from the use of the SOR iterative procedure.

The course followed in evaluating second-order schemes—whose structure is now based on the usual non-coupled (p, q) approach—is similar to that associated with fourth-order schemes, except that very little space is devoted to presenting the structure of the model itself. The reason for this is that such a description would be almost a repetition of the corresponding section concerning the fourth-order schemes. Again numerical experiments for $(p, q) = (1, 2)$ are conducted using both the quasi-direct method and SOR to check the second-order accuracy of these schemes. The reason for selecting the values 1 and 2 for p and q, respectively, was that this particular combination of p and q parameters is associated with the most accurate results available—see [5, 6]—for second-order schemes.

As a final remark on the compact formation of the problem one might note the combination of this with the capacitance matrix approach presented in [3]. However, for both the non-coupled (p, q, r, s) and (p, q) methods the structure of the capacitance matrix is rather complicated, while the number of unknown parameters which must be determined is much larger than the corresponding number in [3]. For this reason this method is not examined in this paper.

2. COMPACT MATRIX FORMULATION OF FOURTH-ORDER SCHEMES

A method for discretizing the first biharmonic problem in a rectangle $a \times b$ to derive computational schemes with fourth-order truncation errors would be to combine the discrete equivalents of $\nabla^2 g = f$ and $\nabla^2 \phi = g$ in a single linear model by approximating the Laplacian through 9-point stencils

$$\begin{matrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{matrix}$$

which have been introduced by Kantorovich and Krylov [8].

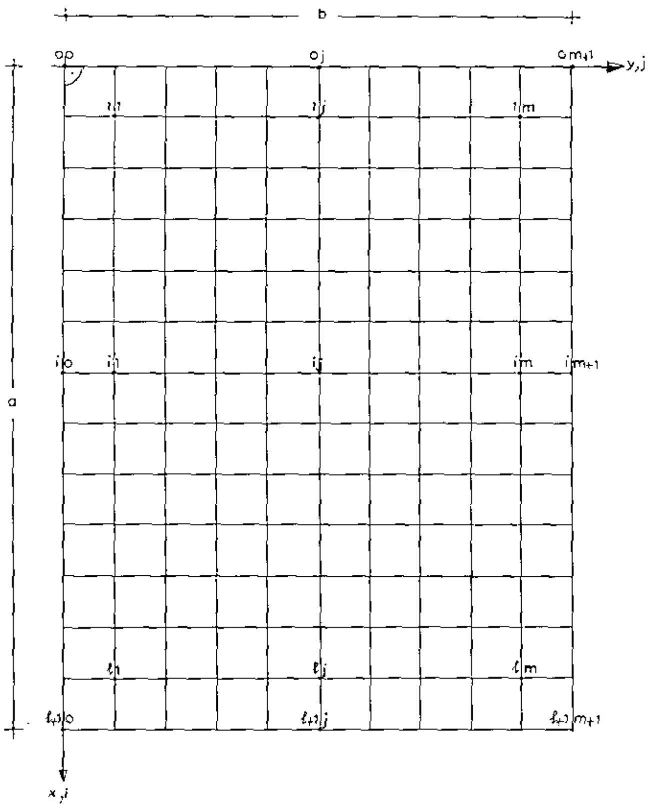

FIGURE 1

The approximations in question are related to a square grid of $l \times m$ nodes with mesh size h equal to the common value of $a/(l+1)$ or $b/(m+1)$, where l and m denote the number of equidistant subdivision points of the edges a and b, respectively, as it is shown in Fig. 1.

Both these problems are considered for the moment of the Dirichlet type (although for $\nabla^2 g = f$ the values of $g = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2$ are not known at the edges). The discrete model of (1), from which the unknown values of ϕ will be calculated, emerges after the corresponding models of $\nabla^2 g = f$ and $\nabla^2\phi = g$ have been combined in such a way that all the boundary conditions are expressed by means of either known or unknown ϕ-values and/or ϕ-derivatives at the edges.

In particular, when searching for the discrete model of $\nabla^2 g = f$ in a rectangle $a \times b$ one must express the values of $g = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2$ at the boundaries as a fourth-order linear combination of either known or unknown ϕ-values and normal ϕ-derivatives at the boundaries in question. Due to the fact that ϕ is known at the boundaries it is obvious that $\partial^2\phi/\partial x^2$ and $\partial^2\phi/\partial y^2$ may be considered as known at ($y = 0$ or $y = b$) and ($x = 0$ or $x = a$), respectively, since they may be expressed as fourth-order linear combinations of known ϕ-values at the corresponding edges [4]. For example, when $y = 0$,

$$(\partial^2\phi/\partial x^2)_{1,o} = [10\phi_{o,o} - 15\phi_{1,o} - 4\phi_{2,o} + 14\phi_{3,o} - 6\phi_{4,o} + \phi_{5,o}]/(12h^2) + O(h^4) \tag{3a}$$

$$(\partial^2\phi/\partial x^2)_{l,o} = [10\phi_{l+1,o} - 15\phi_{l,o} - 4\phi_{l-1,o} + 14\phi_{l-2,o} - 6\phi_{l-3,o} + \phi_{l-4,o}]/(12h^2) + O(h^4) \tag{3b}$$

$$(\partial^2\phi/\partial x^2)_{i,o} = [-\phi_{i-2,o} + 16\phi_{i-1,o} - 30\phi_{i,o} + 16\phi_{i+1,o} - \phi_{i+2,o}]/(12h^2) + O(h^4) \quad \text{for } i = 2(1)l-1 \tag{3c}$$

In other words, one should determine fourth-order approximations for $\partial^2\phi/\partial y^2$ and $\partial^2\phi/\partial x^2$ at the boundaries ($y = 0$ or $y = b$) and ($x = 0$ or $x = a$), respectively, by means of the unknown ϕ-values and the prescribed normal ϕ-derivatives.

Without loss of generality, such an approximation is given for $(\partial^2\phi/\partial y^2)_{i,o}$ at the edge $y = 0$ for $i = 1(1)l$. This approximation, which uses the values of ϕ at the four nodes (i, p), (i, q), (i, r), and (i, s) and whose derivation is given in the Appendix, is the following:

$$(\partial^2\phi/\partial y^2)_{i,o} = [-(\beta_p + \beta_q + \beta_r + \beta_s)\phi_{i,o} + \beta_p\phi_{i,p} + \beta_q\phi_{i,q} + \beta_r\phi_{i,r} + \beta_s\phi_{i,s} - (p\beta_p + q\beta_q + r\beta_r + s\beta_s)h(\partial\phi/\partial y)_{i,o}]/[0.5h^2(p^2\beta_p + q^2\beta_q + r^2\beta_r + s^2\beta_s)] + O(h^4)$$

with

$$\begin{aligned} \beta_p &= (srq)^2(s-r)(s-q)(r-q) \\ \beta_q &= -(srp)^3(s-r)(s-p)(r-p) \\ \beta_r &= (sqp)^3(s-q)(s-p)(q-p) \\ \beta_s &= -(rqp)^3(r-q)(r-p)(q-p). \end{aligned} \tag{4}$$

For $(p, q, r, s) = (1, 2, 3, 4)$ the latest relation (4)—after the calculated coefficients β_p, β_q, β_r, and β_s have been divided by 48—takes the form,

$$(\partial^2\phi/\partial y^2)_{i,o} = [-415\phi_{i,o} + 576\phi_{i,p} - 216\phi_{i,q} + 64\phi_{i,r} - 9\phi_{i,s} - 300h(\partial\phi/\partial y)_{i,o}]/(72h^2) + O(h^4). \tag{4a}$$

The first step in constructing the discrete model of $\nabla^2 g = f$—when $(p, q, r, s) = (1, 2, 3, 4)$—is to present the approximate forms of the p.d.e., both at an interior grid node ij and at one boundary node, e.g., at the i,o node of the edge $y = 0$. Both of these discrete equations have fourth-order truncation errors and run as—see also Eq. (10.3) in [2],

Interior node ij:

$$(1/6h^2)[g_{i-1,j-1} + 4g_{i-1,j} + g_{i-1,j+1} + 4g_{i,j-1} - 20g_{ij} + 4g_{i,j+1} + g_{i+1,j-1} + 4g_{i+1,j} + g_{i+1,j+1}] = \nabla^2 g_{ij} + (h^2/12)\nabla^4 g_{ij} + O(h^4)$$

or, since $\nabla^2 g_{ij} = f_{ij}$,

$$(1/6h^2)[g_{i-1,j-1} + 4g_{i-1,j} + g_{i-1,j+1} + 4g_{i,j-1} - 20g_{ij} + 4g_{i,j+1} + g_{i+1,j-1} + 4g_{i+1,j} + g_{i+1,j+1}] = f_{ij} + (h^2/12) = \nabla^2 f_{ij} + O(h^4) \quad (5)$$

Boundary node i,o at the edge $y = 0$:

$$g_{i,o} = (\partial^2\phi/\partial x^2)_{i,o} + [-415\phi_{i,o} + 576\phi_{i,1} - 216\phi_{i,2} + 64\phi_{i,3} - 9\phi_{i,4} - 300h(\partial\phi/\partial y)_{i,o}]/(72h^2) + O(h^4) \quad (5a)$$

In the latest relation (5a) the quantity $(\partial^2\phi/\partial x^2)_{i,o}$ is considered as known and is given by one of the formulas (3a), (3b), or (3c) when $i = 1$, $i = l$, and $i = 2(1)l - 1$, respectively. Obviously, apart from (5a) there are three other similar types of relations corresponding to the boundary nodes, $(i, m + 1)$, (o, j), and $(l + 1, j)$, at the edges $y = b$, $x = 0$, and $x = a$, respectively. It is possible to express the $l \times m$ relations (5) by means of a compact matrix equation in which the discrete expressions for $g = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2$ at the boundary as given, e.g., by (5a), are also taken into account. Towards this end, a number of matrices needed to formulate this compact matrix equation must be introduced. A first group of such matrices is the following:

F, G = Two $l \times m$ matrices composed of the values of the source term f and $g = \nabla^2\phi$ at the interior points of the grid, respectively. (6)

G_{lr} = An $l \times m$ matrix whose first and last columns only may be composed of non-zero elements. In particular, the first and last columns contain the values of $g_{i,o}$ at the edge $y = 0$ and the values of $g_{i,m+1}$ at the edge $y = b$, respectively ($i = 1, 2, ..., l$). (7)

G_{ud} = An $l \times m$ matrix whose first and last rows only may be composed of non-zero elements. The first and last rows contain the values of $g_{o,j}$ at the edge $x = 0$ and $g_{l+1,j}$ at $x = a$, respectively ($j = 1, 2, ..., m$). (8)

G_c = An $l \times m$ matrix whose four "corner" elements only may be different from zero. In particular,

$$(G_c)_{1,1} = g_{o,o} = \nabla^2\phi_{o,o}$$
$$(G_c)_{1,m} = g_{o,m+1} = \nabla^2\phi_{o,m+1}$$
$$(G_c)_{l,m} = g_{l+1,m+1} = \nabla^2\phi_{l+1,m+1}$$
$$(G_c)_{l,1} = g_{l+1,o} = \nabla^2\phi_{l+1,o}.$$

Since these four elements may be expressed as fourth-order linear combinations of ϕ-values on the boundary after using relations similar to (3a), (3b), and (3c), which are also cited in [4], they are considered as known quantities. (9)

$\nabla^2 F$ = An $l \times m$ matrix composed of the known values of $\nabla^2 f_{ij}$ at the interior nodes of the grid. Since the subsequent analysis will make it evident that to construct a fourth-order discrete model of (1) it suffices to express $\nabla^2 f_{ij}$ as a second-order linear combination of f-values, one may use the expression

$$\nabla^2 f_{ij} = (1/6h^2)[f_{i-1,j-1} + 4f_{i-1,j} + f_{i-1,j+1} + 4f_{i,j-1} - 20f_{ij} + 4f_{i,j+1} + f_{i+1,j-1} + 4f_{i+1,j} + f_{i+1,j+1}] \quad (10)$$

$A_\nu(\alpha, \beta)$ = A symmetric tridiagonal matrix of order ν with all its diagonal elements equal to α and all its off-diagonal ones equal to β. With respect to the above matrix, it is known that the ijth element of the orthonormal matrix V_ν of its normalized eigenvectors is equal to $[2/(\nu + 1)]^{1/2} \sin[ij\pi/(\nu + 1)]$ while its jth eigenvalue equals $\alpha + 2\beta\cos[j\pi/(\nu + 1)]$ with $i = 1(1)\nu$ and $j = 1(1)\nu$. (11)

KANKR[ ] = A matrix operator which acts on the $l \times m$ matrix [ ] and is defined as follows: KANKR[ ] = $A_l(-10, 4)$[ ] + $A_l(0, 1)$[ ]$A_m(0, 1)$ + [ ]$A_m(-10, 4)$. (12)

By means of the above notation, (6) to (12), it is possible to express the set of all the $l \times m$ difference equations (5) as a single matrix equation by employing the procedure presented in [9], which in effect constitutes an extension of the so-called "irrational method" [1]. In this way the following equation emerges:

$$\text{KANKR}[G] + A_l(4, 1)G_{lr} + G_{ud}A_m(4, 1) + G_c = (6h^2)[F + (1/12)h^2\nabla^2 F + O(h^4)]. \quad (13)$$

This is not the final form of the matrix equation for the discrete model of $\nabla^2 g = f$, since the elements of G_{lr} and G_{ud} may be further expressed by means of the second ϕ-derivatives and the given first normal derivatives at the boundary, as well as the unknown ϕ-values themselves.

Presentation of these expressions in compact form requires the introduction of a second group of matrices as follows:

$SD_{x,lr}$ = An $l \times m$ matrix with non-zero elements only at its first and last columns. The elements of the first column are the (usually approximated by fourth-order combinations of boundary ϕ-values) second derivatives $(\partial^2\phi/\partial x^2)_{i,o}$ at $y = 0$, while the elements of the mth column are the corresponding quantities $(\partial^2\phi/\partial x^2)_{i,m+1}$ at the edge $y = b$ $(i = 1, 2, ..., l)$. (14)

$SD_{y,ud}$ = An $l \times m$ matrix with non-zero elements only at the first and last rows, which contain the (usually approximated by fourth-order combinations of boundary ϕ-values) second derivatives $(\partial^2\phi/\partial y^2)_{o,j}$ and $(\partial^2\phi/\partial y^2)_{l+1,j}$ at the edges $x = 0$ and $x = a$, respectively $(j = 1, 2, ..., m)$. (15)

$FD_{y,lr}$ = An $l \times m$ matrix with non-zero elements only at the first and last columns. In particular, the first column contains the given values of $+(\partial\phi/\partial y)_{i,o}$ at $y = 0$ while the last column has the given values of $-(\partial\phi/\partial y)_{i,m+1}$ at $y = b$ $(i = 1, 2, ..., l)$. (16)

$FD_{x,ud}$ = An $l \times m$ matrix with non-zero elements only at its first and last rows, which contain the given values of $+(\partial\phi/\partial x)_{o,j}$ at $x = 0$ and $-(\partial\phi/\partial x)_{l+1,j}$ at $x = a$, respectively $(j = 1, 2, ..., m)$. (17)

Φ, Φ_{lr}, Φ_{ud} = Three $l \times m$ matrices referring to ϕ-values whose meaning is similar to that of the matrices G, see (6), G_{lr}, see (7), and G_{ud}, see (8), respectively. As a matter of fact, the verbal description of these Φ-matrices is almost identical to the description of the corresponding G-matrix, provided of course that the parameter $g = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2$ has been replaced by ϕ. (18)

N_ν = A square matrix of order ν, such that only the first four elements of its first row and the last four of its last row are different from zero. The values of these elements are equal to a multiple of the coefficients multiplying the four unknown ϕ-values in the expression giving the second normal derivative of ϕ at the boundary; see, e.g., (4) or (4a)—in particular, for $(p, q, r, s) = (1, 2, 3, 4)$:

$$\begin{aligned}(N_\nu)_{1,1} &= (N_\nu)_{\nu,\nu} = 576\\ (N_\nu)_{1,2} &= (N_\nu)_{\nu,\nu-1} = -216\\ (N_\nu)_{1,3} &= (N_\nu)_{\nu,\nu-2} = 64\\ (N_\nu)_{1,4} &= (N_\nu)_{\nu,\nu-3} = -9.\end{aligned} \tag{19}$$

Using notations (14) to (19) it is possible to express G_{lr} and G_{ud} in terms of $SD_{x,lr}$, $SD_{y,ud}$, $FD_{y,lr}$, $FD_{x,ud}$, Φ, Φ_{lr}, and Φ_{ud}. These expressions, which constitute in effect the compact form of relations of the types (3a), (3b), (3c), and (4a), take the form (′ stands for the transposed of a matrix):

$$G_{lr} = SD_{x,lr} + (1/72h^2)[-415\Phi_{lr} - 300h \cdot FD_{y,lr} + \Phi N_m'] + O(h^4) \tag{20}$$

$$G_{ud} = SD_{y,ud} + (1/72h^2)[-415\Phi_{ud} - 300h \cdot FD_{x,ud} + N_l\Phi] + O(h^4). \tag{21}$$

The discrete model of $\nabla^2 g = f$ is now formed after replacing G_{lr} and G_{ud} in (13) by their equivalent expressions (20) and (21), respectively. The result of such a replacement is

$$\begin{aligned}&\text{KANKR}[G] + (1/72h^2)[A_l(4,1)\Phi N_m' + N_l\Phi A_m(4,1)]\\ &\quad = (6h^2)[F + (1/12)h^2\nabla^2 F - G_c + O(h^4)] - [A_l(4,1)SD_{x,lr}\\ &\quad + SD_{y,ud}A_m(4,1)]\\ &\quad + (1/72h^2)A_l(4,1)\cdot[415\Phi_{lr} + 300h \cdot FD_{y,lr}]\\ &\quad + (1/72h^2)[415\Phi_{ud} + 300h \cdot FD_{x,ud}]\cdot A_m(4,1) + O(h^4).\end{aligned} \tag{22}$$

The next step in constructing the discrete model of (1) is to form the discrete equivalent of $\nabla^2\phi = g$ and combine this with (22). The only additional matrix which must be introduced in analogy to G_c—see (9)—is now the $l \times m$ matrix Φ_c whose four "corner" elements only are different from zero and are given as

$$\{(\Phi_c)_{1,1}, (\Phi_c)_{1,m}, (\Phi_c)_{l,m}, (\Phi_c)_{l,1}\} = \{\phi_{o,o}, \phi_{o,m+1}, \phi_{l+1,m+1}, \phi_{l+1,o}\}. \tag{23}$$

Again employing Eq. (10.3) of [2] and keeping in mind that $\nabla^2\phi = g$ and $\nabla^4\phi = f$, it is possible to express the fourth-order discrete equivalent of $\nabla^2\phi = g$ as

$$\text{KANKR}[\Phi] + A_l(4,1)\Phi_{lr} + \Phi_{ud}A_m(4,1) + \Phi_c = (6h^2)[G + (1/12)h^2 F + O(h^4)] \tag{24}$$

Finally, the elimination of the $l \times m$ matrix G from (22) and (24) leads, after a rather tedious but straightforward procedure, to the relation (25) which expresses the fourth-order discrete equivalent of (1), as far as the first biharmonic problem in a rectangle $a \times b$ is concerned:

$$\begin{aligned}&\mathrm{KANKR}[\mathrm{KANKR}[\Phi]] + (1/12)[A_l(4,1)\Phi N'_m + N_l\Phi A_m(4,1)]\\ &\quad = 36h^4[F + (h^2/12)\nabla^2 F - G_c] + \mathrm{KANKR}[(1/2)h^4 F\\ &\quad - A_l(4,1)\Phi_{lr} + \Phi_{ud}A_m(4,1)] - [A_l(4,1)\cdot(6h^2)SD_{x,lr}\\ &\quad + (6h^2)SD_{y,ud}\cdot A_m(4,1)]\\ &\quad + (1/12)A_l(4,1)\cdot[415\Phi_{lr} + 300h\cdot FD_{y,lr}]\\ &\quad + (1/12)[415\Phi_{ud} + 300h\cdot FD_{x,ud}]\cdot A_m(4,1).\end{aligned}$$

Setting KANKR[KANKR[ ]] = KANKR^2[ ] and denoting the right-hand side of (25) as F_{kankr} the fourth-order discrete equivalent of (1) takes the form

$$\mathrm{KANKR}^2[\Phi] + (1/12)[A_l(4,1)\Phi N'_m + N_l\Phi A_m(4,1)] = F_{\mathrm{kankr}}. \tag{26}$$

3. THE TRANSFORMED MATRIX EQUATION FOR FOURTH-ORDER SCHEMES

Although (26) is the final discrete model for (1) concerning the first biharmonic problem in rectangles, the unknown $l \times m$ matrix Φ will not be directly computed from this model but through the computation of another $l \times m$ matrix X whose elements are linearly dependent on those of Φ.

Toward this end (26) is first premultiplied and then post-multiplied by the matrices V_l and V_m—see (11)—respectively, resulting in

$$\begin{aligned}&V_lA_l(-10,4)[A_l(-10,4)\Phi + A_l(0,1)\Phi A_m(0,1)\\ &\quad + \Phi A_m(-10,4)]V_m\\ &\quad + V_lA_l(0,1)[A_l(-10,4)\Phi\\ &\quad + A_l(0,1)\Phi A_m(0,1) + \Phi A_m(0,1)]A_m(0,1)V_m\\ &\quad + V_l[A_l(-10,4)\Phi + A_l(0,1)\Phi A_m(0,1) + \Phi A_m(-10,4)\\ &\quad + \Phi A_m(-10,4)]V_m + (1/12)V_l[A_l(4,1)\Phi N'_m\\ &\quad + N_l\Phi A(4,1)]V_m = V_lF_{\mathrm{kankr}}V_m.\end{aligned}$$

Since both V_l and V_m are orthonormal, the above matrix equation can be equivalently written as

$$\begin{aligned}&V_lA_l(-10,4)V_l \times [V_lA_l(-10,4)V_l(V_l\Phi V_m)\\ &\quad + V_lA_l(0,1)V_l(V_l\Phi V_m)V_mA_m(0,1)V_m\\ &\quad + (V_l\Phi V_m)V_mA_m(-10,4)V_m]\\ &\quad + V_lA_l(0,1)V_l \times [V_lA_l(-10,4)V_l(V_l\Phi V_m)\\ &\quad + V_lA_l(0,1)V_l(V_l\Phi V_m)V_mA_m(0,1)V_m\\ &\quad + (V_l(\Phi V_m)V_mA_m(-10,4)V_m] \times V_mA_m(0,1)V_m\\ &\quad + [V_lA_l(-10,4)V_l(V_l\Phi V_m)\\ &\quad + V_lA_l(0,1)V_l(V_l\Phi V_m)V_mA_m(0,1)V_m\\ &\quad + (V_l\Phi V_m)V_mA_m(-10,4)V_m] \times V_mA_m(-10,4)V_m\\ &\quad + (1/12)[V_lA_l(4,1)V_l(V_l\Phi V_m)V_mN'_mV_m]\\ &\quad + (1/12)[3V_lN_lV_l(V_l\Phi V_m)V_mA_m(4,1)V_m] = V_lF_{\mathrm{kankr}}V_m.\end{aligned} \tag{28}$$

However, due to the fact that $V_\nu A_\nu(\alpha, \beta)V_\nu$ is in effect, the diagonal matrix $D_\nu(\alpha, \beta)$ of the eigenvalues $\alpha + 2\beta \cos[j\pi/(\nu + 1)]$ $(j = 1, 2, \ldots, \nu)$ of the $\nu \times \nu$ matrix $A_\nu(\alpha, \beta)$, as it is apparent from (11), the matrix equation (28) can be written using the additional notation

$$V_l\Phi V_m = Y \tag{29}$$

as

$$\begin{aligned}&D_l(-10,4)[D_l(-10,4)Y + D_l(0,1)YD_m(0,1)\\ &\quad + YD_m(-10,4)] + D_l(0,1)[D_l(-10,4)Y\\ &\quad + D_l(0,1)YD_m(0,1) + YD_m(-10,4)]D_m(0,1)\\ &\quad + [D_l(-10,4)Y + D_l(0,1)YD_m(0,1)\\ &\quad + YD_m(-10,4)]D_m(-10,4)\\ &\quad + (1/12)[D_l(4,1)YV_mN'_mV_m\\ &\quad + V_lN_lV_lD_m(4,1)] = V_lF_{\mathrm{kankr}}V_m.\end{aligned} \tag{30}$$

In the above equation the two matrix products $V_mN'_mV_m$ and $V_lN_lV_l$ may be written in a way that will help in providing guidlines for a quasi-direct method for the solution of the final transformed version of the discrete model of the problem.

When examining the form of the $\nu x \nu$ matrix $V_\nu N'_\nu V_\nu$—with N_ν given by (19)—for $(p, q, r, s) = (1, 2, 3, 4)$, one should keep in mind that the elements i, j and $i,(\nu - j + 1)$ of the symmetric $\nu x \nu$ matrix V_ν are equal to $[2/(\nu + 1)]^{1/2} \sin[ij\pi/(\nu + 1)]$ and $(-1)^{i-1}[2/(\nu + 1)]^{1/2} \sin[ij\pi/(\nu + 1)]$, respectively. Using this information one can prove in a direct fashion that

$$\begin{aligned}(V_\nu N'_\nu N_\nu)_{ij} = 2[576(V_\nu)_{i,1} - 216(V_\nu)_{i,2} + 64(V_\nu)_{i,3}\\ - 9(V_\nu)_{i,4}]\frac{1 + (-1)^{i+j-2}}{2}(V_\nu)_{1,j}.\end{aligned} \tag{31}$$

By means of the symbolism,

E_ν = A diagonal matrix of order ν with elements equal to $(1/12)[1152(V_\nu)_{i,1} - 432(V_\nu)_{i,2} + 128(V_\nu)_{i,3} - 18(V_\nu)_{i,4}]$ $(i = 1, 2, \ldots, \nu)$; (32)

Q_ν = A symmetric $\nu x \nu$ matrix such that $(Q_\nu)_{ij} = (1 + (-1)^{i+j-2})/2$; i.e., a matrix such that only the ij elements with i and j of the same parity are non-zero and equal to 1; (33)

S_ν = a diagonal matrix of order ν with elements equal to $(V_\nu)_{1,j}$ $(j = 1, 2, \ldots, \nu)$; (34)

one can see that relation (31) is equivalent to

$$(V_\nu N_\nu V_\nu)' = V_\nu N'_\nu V_\nu = 12E_\nu Q_\nu S_\nu. \tag{35}$$

The replacement of $V_m N'_m V_m$ and $V_l N_l V_l$ in (30) by their equivalent expressions $12E_m Q_m S_m$ and $12S_l Q_l E_l$, respectively, results in

$$\begin{aligned} D_l(-10,4)[D_l(-10,4)Y + D_l(0,1)YD_m(0,1) \\ + YD_m(-10,4)] \\ + D_l(0,1)[D_l(-10,4)Y + D_l(0,1)YD_m(0,1) \\ + YD_m(-10,4)]D_m(0,1) + [D_l(-10,4)Y \\ + D_l(0,1)YD_m(0,1) \\ + YD_m(-10,4)]D_m(-10,4) \\ + [D_l(4,1)YE_m Q_m S_m \\ + S_l Q_l E_l YD_m(4,1)] = V_l F_{\text{kankr}} V_m. \end{aligned} \tag{36}$$

At this point one may observe that both the diagonal matrices E_ν and S_ν are composed of positive elements when $(p, q, r, s) = (1, 2, 3, 4)$. For S_ν this is obvious, due to the fact that $\sin[j\pi/(\nu + 1)]$ is positive for $j = 1, 2, \ldots, \nu$. As far as the matrix E_ν is concerned, it is noted that due to (11) and (32) its ith element equals a positive multiple of the expression $R = 576 \sin\theta - 216 \sin 2\theta + 64 \sin 3\theta - 9 \sin 4\theta$ with $\theta = i\pi/(\nu + 1)$ $(i = 1, 2, \ldots, \nu)$. Using elementary trigonometry one may replace $\sin 2\theta$, $\sin 3\theta$, and $\sin 4\theta$ in R by $2 \sin\theta \cos\theta$, $\sin\theta(4\cos^2\theta - 1)$, and $4 \sin\theta \cos\theta(2\cos^2\theta - 1)$, respectively. Such a replacement leads, after some elementary algebraic transformations to the relation $R = 4 \sin\theta \cdot [29 + 46 \cos^2\theta + 9(1 - \cos\theta)(11 + 2\cos^2\theta)] > 0$. (Q.E.D.)

Aiming next at a more compact expression for the final discrete model of the original p.d.e. (1), one introduces, by means of relations (37) to (41), a number of matrices which will help to express it concisely:

$$Y = E_l^{-1} X E_m^{-1}; \tag{37}$$

$$\text{DKANKR}[\;] = D_l(-10,4)[\;] + D_l(0,1)[\;]D_m(0,1) + [\;]D_m(-10,4) \quad \text{with } [\;] \text{ an } lxm \text{ matrix};$$

$$E_\nu S_\nu = Z_\nu, \quad \text{a diagonal } \nu x \nu \text{ matrix with positive entries for } (p, q, r, s) = (1, 2, 3, 4); \tag{38}$$

$$D_\nu(4,1) E_\nu^{-1} S_\nu^{-1} = T_\nu, \quad \text{a diagonal } \nu x \nu \text{ matrix with} \tag{39}$$

$$\text{positive entries for } (p, q, r, s) = (1, 2, 3, 4); \tag{40}$$

$$S_l^{-1} V_l F_{\text{kankr}} V_m S_m^{-1} = \text{RHS}. \tag{41}$$

Relation (36) is now premultiplied and postmultiplied by the diagonal matrices S_l^{-1} and S_m^{-1}, respectively. The result of such an operation, after taking into account (37) to (41) and, in addition, the fact that both E_l and $D_l(\alpha, \beta)$, as well as E_m and $D_m(\alpha, \beta)$ commute, is

$$Z_l^{-1} \cdot \text{DKANKR}^2[X] \cdot Z_m^{-1} + T_l X Q_m + Q_l X T_m = \text{RHS}. \tag{42}$$

The latest relation constitutes the final version of the transformed discrete model of (1) when $(p, q, r, s) = (1, 2, 3, 4)$ whose solution X will eventually lead to the unknown lxm matrix Φ by means of

$$\Phi = V_l E_l^{-1} X E_m^{-1} V_m, \tag{43}$$

after combining (29) and (37).

Apart from its compact expression (42) the transformed model of (1) can be written conventionally as a product of an $(lxm) \times (lxm)$ coefficient matrix premultiplying an $(lxm) \times 1$ vector of unknowns equal to an $(lxm) \times 1$ vector of known quantities.

Assuming that both the $(lxm) \times 1$ vectors of unknown and known quantities are composed of the m consecutive columns of the lxm matrices X and the RHS, respectively, the $(lxm) \times (lxm)$ matrix of coefficients A^* (the "big matrix" as it is called in [1]) is given as

$$\begin{aligned} A^* = (Z_m^{-1} \otimes Z_l^{-1}) \cdot [I_m \otimes D_l(-10,4) + D_m(0,1) \otimes D_l(0,1) \\ + D_m(-10,4) \otimes I_l]^2 + Q_m \otimes T_l + T_m \otimes Q_l \end{aligned} \tag{44}$$

or

$$\begin{aligned} A^* = (Z_m^{-1} \otimes Z_l^{-1})[I_m \otimes D_l(-10,4) + D_m(0,1) \otimes D_l(0,1) \\ + D_m(-10,4) \otimes I_l]^2 + I_m \otimes T_l \\ + T_m \otimes I_l + (Q_m - I_m) \otimes T_l + T_m \otimes (Q_l - I_l) \end{aligned} \tag{44a}$$

with I_ν and $\otimes$ standing for the unit matrix of order ν and the symbol for the tensor product, respectively.

Due to the structure of (44a) it is evident that A^* is a symmetric $(lxm) \times (lxm)$ matrix composed of a purely diagonal part,

$$\begin{aligned} (Z_m^{-1} \otimes Z_l^{-1})[I_m \otimes D_l(-10,4) + D_m(0,1) \otimes D_l(0,1) \\ + D_m(-10,4) \otimes I_l]^2 + I_m \otimes T_l + T_m \otimes I_l, \end{aligned}$$

and a symmetric complementary part, $(Q_m - I_m) \otimes T_l + T_m \otimes (Q_l - I_l)$, with zero diagonal entries.

Since it has already been proved that the elements of the matrices S_ν and E_ν are positive and that the same holds for those of $D_\nu(4, 1)$—i.e., for the quantities $4 + 2\cos[j\pi/(\nu + 1)]$—it follows that the elements of the diagonal matrices $Z_\nu = E_\nu S_\nu$ and $T_\nu = D_\nu(4,1) E_\nu^{-1} S_\nu^{-1}$ are positive as well. Consequently the lxm entries c_{ij} (for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, l$) of the purely diagonal part of A^*, namely,

$$\begin{aligned} c_{ij} = (Z_l^{-1})_i \{[D_l(-10,4)]_i + [D_l(0,1)]_i [D_m(0,1)]_j \\ + [D_m(-10,4)]_j\}^2 (Z_m^{-1})_j + (T_l)_i + (T_m)_j; \end{aligned} \tag{45}$$

i.e., $c_{ij} = d_{ij} + (T_l)_i + (T_m)_j$ with

$$\begin{aligned} d_{ij} = (Z_l^{-1})_i \{[D_l(-10,4)]_i + [D_l(0,1)]_i [D_m(0,1)]_j \\ + [D_m(-10,4)]_j\}^2 (Z_m^{-1})_j \end{aligned} \tag{45a}$$

are positive as well.

The fact that the transformed discrete model (42) when expressed in conventional form is equivalent to a symmetric linear system, with a coefficient matrix having positive diagonal entries, might suggest the employment of the SOR method to solve it.

As a matter of fact, according to the Ostrowski–Reich theorem, convergence of the SOR iterative scheme is certain when, in addition to the above characteristics, A* is also positive definite [2]. Ignoring for the moment the spectral characteristics of A*, such an iterative scheme for the model (42) may be expressed as $(1 < \omega < 2)$

Initial stage:

$$\overset{(o)}{X}_{ij} = 0 \quad (\text{for } j = 1, 2, \ldots, m;\ i = 1, 2, \ldots, l) \tag{46a}$$

Stage $(n + 1)$:

$$\begin{aligned}\overset{(n+1)}{X}_{ij} = (1-\omega)\overset{(n)}{X}_{ij} + \omega\Bigg\{&(\mathrm{RHS})_{ij} \\ &- \left[\sum_{k=1}^{i-1}(Q_l)_{ik}\overset{(n+1)}{X}_{kj} + \sum_{k=i+1}^{l}(Q_l)_{ik}\overset{(n)}{X}_{kj}\right]\cdot(T_m)_j \\ &- (T_l)_i\cdot\left[\sum_{k=1}^{j-1}\overset{(n+1)}{X}_{ik}(Q_m)_{kj} + \sum_{k=j+1}^{m}\overset{(n)}{X}_{ik}(Q_m)_{kj}\right]\Bigg\}\Big/c_{ij}\end{aligned} \tag{46b}$$

$(\text{for } j = 1, 2, \ldots, m;\ i = 1, 2, \ldots, l)$.

Turning now to the assumption of the positive definitness of A*, it is noted that such an assumption seems more or less reasonable, since any eigenvalue of (42)—which obviously coincides with an eigenvalue of the source model (26)—may be identified with a positive multiple of one of the natural frequencies of a plate with all its edges clamped. Anyhow, irrespective of the validity of the above assumption, the numerical experiments showed convergence in all cases where the SOR method was applied. Apart from applying the SOR method, there is also the possibility of solving (42) by means of a quasi-direct method as described in the next section.

4. SOLUTION OF THE TRANSFORMED MATRIX EQUATION FOR FOURTH-ORDER SCHEMES BY A QUASI-DIRECT METHOD

Before attempting a detailed presentation of a quasi-direct method for the solution of the discrete model (42), it seems reasonable to examine the structure of the model itself, since this may be of help in suggesting some key features of the method in question. In this context one should have in mind the special role played by the two matrix products Q_lX and XQ_m. In order to clarify such a role one first defines k_{odd} and k_{even} as the number of odd and even numbered elements, respectively, in the set $\{1, 2, 3, \ldots, k-2, k-1, k\}$ and satisfying the obvious condition that $k_{\text{odd}} + k_{\text{even}} = k$. As far as the structure of the matrix Q_lX is concerned, it is noted that all the jth elements of all the odd-numbered rows of it are equal to $\sum_{\nu=1}^{l_{\text{odd}}} x_{2\nu-1,j}$, while all the jth elements of its even-numbered rows equal $\sum_{\nu=1}^{l_{\text{even}}} x_{2\nu,j}$. Similar remarks hold for XQ_m in which all the ith elements of its odd-numbered columns are equal to $\sum_{\nu=1}^{m_{\text{odd}}} x_{i,2\nu-1}$ and all the ith elements of the even-numbered columns of it are equal to $\sum_{\nu=1}^{m_{\text{even}}} x_{i,2\nu}$.

Next, the symbolism

$$C_j^o = \left(\sum_{\nu=1}^{l_{\text{odd}}} x_{2\nu-1,j}\right)\cdot(T_m)_j \quad (j = 1, 2, \ldots, m) \tag{47}$$

$$C_j^e = \left(\sum_{\nu=1}^{l_{\text{even}}} x_{2\nu,j}\right)\cdot(T_m)_j \quad (j = 1, 2, \ldots, m) \tag{47a}$$

$$R_i^o = (T_l)_i\cdot\left(\sum_{\nu=1}^{m_{\text{odd}}} x_{i,2\nu-1}\right) \quad (i = 1, 2, \ldots, l) \tag{48}$$

$$R_i^e = (T_l)_i\cdot\left(\sum_{\nu=1}^{m_{\text{even}}} x_{i,2\nu}\right) \quad (i = 1, 2, \ldots, l) \tag{48a}$$

is introduced which, combined with (45a), helps to express (42) in the form

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \ldots d_{ij}X_{ij}\ldots \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} + \underset{(Q_lXT_m)}{\begin{bmatrix} C_1^o & \ldots & C_j^o & \ldots & C_m^o \\ C_1^e & \ldots & C_j^e & \ldots & C_m^e \\ \cdot & \ldots & \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot & \ldots & \cdot \end{bmatrix}} + \underset{(T_lXQ_m)}{\begin{bmatrix} R_1^o & R_1^e & \ldots \\ \cdot & \cdot & \ldots \\ R_i^o & R_i^e & \ldots \\ \cdot & \cdot & \ldots \\ R_l^o & R_l^e & \ldots \end{bmatrix}} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \ldots(\mathrm{RHS})_{ij}\ldots \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}. \tag{49}$$

A direct consequence of the latest equation is the expression of X in terms of the $2(l + m)$ unknown parameters (C_j^o, C_j^e) and (R_i^o, R_i^e) for $j = 1(1)m$ and $i = 1(1)l$.

More precisely, it is easily seen that (49) may be equivalently written in the form

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \ldots X_{ij} \ldots \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} + \begin{bmatrix} d_{11}^{-1}C_1^o \ldots d_{1j}^{-1}C_j^o \ldots d_{1m}^{-1}C_m^o \\ d_{21}^{-1}C_1^e \ldots d_{2j}^{-1}C_j^e \ldots d_{2m}^{-1}C_m^e \\ \cdot \ \ldots \ \cdot \ \ldots \ \cdot \\ \cdot \ \ldots \ \cdot \ \ldots \ \cdot \\ \cdot \ \ldots \ \cdot \ \ldots \ \cdot \\ \cdot \ \ldots \ \cdot \ \ldots \ \cdot \\ \cdot \ \ldots \ \cdot \ \ldots \ \cdot \end{bmatrix}$$

$$+ \begin{bmatrix} d_{11}^{-1}R_1^o & d_{12}^{-1}R_1^e & \ldots \\ d_{21}^{-1}R_2^o & d_{22}^{-1}R_2^e & \ldots \\ \vdots & \vdots & \ldots \\ d_{i1}^{-1}R_i^o & d_{i2}^{-1}R_i^e & \ldots \\ \vdots & \vdots & \ldots \\ d_{l1}^{-1}R_l^o & d_{l2}^{-1}R_l^e & \ldots \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \ldots d_{ij}^{-1}(\text{RHS})_{ij} \ldots \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}. \quad (50)$$

The latest matrix equation will play a key role in producing necessary conditions which will eventually lead to the calculation of the $2(l + m)$ unknown parameters (C_j^o, C_j^e) and (R_i^o, R_i^e).

Towards this end, X is written as $(XT_m)T_m^{-1}$ and (50) is premultiplied by Q_l. This premultiplication results in the following matrix equation (51) in which for each participating matrix, all the odd-numbered rows of it are identical to its first row, while all its even-numbered ones are identical to its second row. For this reason in presenting (51) only the first two rows of the matrices of the left-hand side are shown:

$$\begin{array}{lcc} \text{Column:} & 2\lambda - 1 & 2\lambda \end{array}$$

$$\begin{bmatrix} \ldots & (T_m^{-1})_{2\lambda-1} \cdot C_{2\lambda-1}^o & (T_m^{-1})_{2\lambda} \cdot C_{2\lambda}^o & \ldots \\ \ldots & (T_m^{-1})_{2\lambda-1} \cdot C_{2\lambda-1}^e & (T_m^{-1})_{2\lambda} \cdot C_{2\lambda}^e & \ldots \end{bmatrix}$$

$$+ \begin{bmatrix} \ldots & \left(\sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\lambda-1}^{-1}\right) \cdot C_{2\lambda-1}^o & \left(\sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\lambda}^{-1}\right) \cdot C_{2\lambda}^o & \ldots \\ \ldots & \left(\sum_{\nu=1}^{l_{\text{even}}} d_{2\nu,2\lambda-1}^{-1}\right) \cdot C_{2\lambda-1}^e & \left(\sum_{\nu=1}^{l_{\text{even}}} d_{2\nu,2\lambda}^{-1}\right) \cdot C_{2\lambda}^e & \ldots \end{bmatrix}$$

$$+ \begin{bmatrix} \ldots & \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\lambda-1}^{-1} \cdot R_{2\nu-1}^o & \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\lambda}^{-1} \cdot R_{2\nu-1}^e & \ldots \\ \ldots & \sum_{\nu=1}^{l_{\text{even}}} d_{2\nu,2\lambda-1}^{-1} \cdot R_{2\nu}^o & \sum_{\nu=1}^{l_{\text{even}}} d_{2\nu,2\lambda}^{-1} \cdot R_{2\nu}^e & \ldots \end{bmatrix}$$

$$= Q_l \cdot [d_{ij}^{-1}(\text{RHS})_{ij}]. \quad (51)$$

By equating the corresponding elements of the two sides of (51), $2m$ independent linear equations are formed relating the $2(l + m)$ unknown parameters (C_j^o, C_j^e) and (R_i^o, R_i^e) for $j = 1(1)m$ and $i = 1(1)l$. In order to express the set of these $2m$ relations in matrix form one introduces the following notation:

D_c^o = a diagonal matrix of order m such that

$$(D_c^o)_j = (T_m^{-1})_j + \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,j}^{-1}; \quad (52)$$

D_c^e = a diagonal matrix of order m such that

$$(D_c^e)_j = (T_m^{-1})_j + \sum_{\nu=1}^{l_{\text{even}}} d_{2\nu,j}^{-1}; \quad (53)$$

A_{oo} = an mxl matrix with non-zero elements only at the intersection of the odd-numbered rows $2k - 1$ and odd-numbered columns $2\lambda - 1$ and such that

$$(A_{oo})_{2\kappa-1,2\lambda-1} = d_{2\lambda-1,2\kappa-1}^{-1} \text{ with } \kappa = 1(1)m_{\text{odd}} \text{ and } \lambda = 1(1)l_{\text{odd}}; \quad (54)$$

A_{eo} = an mxl matrix with non-zero elements only at the intersection of the even-numbered rows 2κ and odd-numbered columns $2\lambda - 1$ and such that

$$(A_{eo})_{2\kappa,2\lambda-1} = d_{2\lambda-1,2\kappa}^{-1} \text{ with } \kappa = 1(1)m_{\text{even}} \text{ and } \lambda = 1(1)l_{\text{odd}}; \quad (55)$$

A_{oe} = an mxl matrix with non-zero elements only at the intersection of the odd-numbered rows $2\kappa - 1$ and even-numbered columns 2λ and such that

$$(A_{oe})_{2\kappa-1,2\lambda} = d_{2\lambda,2\kappa-1}^{-1} \text{ with } \kappa = 1(1)m_{\text{odd}} \text{ and } \lambda = 1(1)l_{\text{even}}; \quad (56)$$

A_{ee} = an mxl matrix with non-zero elements only at the intersection of the even-numbered rows 2κ and even-numbered columns 2λ and such that

$$(A_{ee})_{2\kappa,2\lambda} = d_{2\lambda,2\kappa}^{-1} \text{ with } \kappa = 1(1)m_{\text{even}} \text{ and } \lambda = 1(1)l_{\text{even}}; \quad (57)$$

U^o, U^e = the $mx1$ vectors of C_j^o's and C_j^e's, respectively; (58)

V^o, V^e = the $lx1$ vectors of R_i^o's and R_i^e's, respectively; (59)

a^o = the $mx1$ vector of the known quantities

$$\sum_{\nu=1}^{l_{odd}} d^{-1}_{2\nu-1,j}\cdot(\text{RSH})_{2\nu-1,j} \quad \text{with } j = 1(1)m; \tag{60}$$

a^e = the $mx1$ vector of the known quantities

$$\sum_{\nu=1}^{l_{even}} d^{-1}_{2\nu,j}\cdot(\text{RSH})_{2\nu,j} \quad \text{with } j = 1(1)m. \tag{61}$$

By means of relations (52) to (61), the matrix equation (51) becomes equivalent to the following linear system:

$$D_c^o U^o + A_{oo}V^o + A_{eo}V^e = a^o \tag{62}$$

$$D_c^e U^e + A_{oe}V^o + A_{ee}V^e = a^e. \tag{63}$$

Before presenting another pair of equations similar to (62) and (63) which, together with the former ones, will lead to the computation of U^o, U^e, V^o, and V^e, one should make some remarks on the structure of the coefficient matrices D_c^o, D_c^e, A_{oo}, A_{eo}, A_{oe}, and A_{ee}.

First the elements of all these matrices—at least for p, q, r, and s equal to 1, 2, 3, and 4, respectively—are positive since they are given by (45a) and the elements of the diagonal matrix T_m^{-1} are also known to be positive. In addition, due to the structure of D_c^o and D_c^e one may conclude that every (diagonal) element of D_c^o is greater than the sum of the elements of A_{oo} and A_{eo} which lie on the same row, while every (diagonal) element of D_c^e is greater than the sum of the elements of the matrices A_{oe} and A_{ee} lying again on the same row with it.

The next step in trying to form the linear system for computing the $2(l + m)$ unknown parameters (C_j^o, C_j^e) and (R_i^o, R_i^e) with $j = 1(1)m$ and $i = 1(1)l$, is to write X as $T_l^{-1}(T_lX)$ and postmultiply (50) by Q_m. The result of this operation is a relation analogous to (51) in which for each participating matrix all the odd-numbered columns of it are identical to its first column, and all its even-numbered columns are identical to its second column.

Employing a similar procedure to the one that resulted in (62) and (63), one may prove that

$$A'_{oo}U^o + A'_{oe}U^e + D_r^oV^o = b^o \tag{64}$$

$$A'_{eo}U^o + A'_{ee}U^e + D_r^eV^e = b^e \tag{65}$$

($'$ stands for the transpose of a matrix).

In the above relations the meaning of the lxl diagonal matrices D_r^o and D_r^e, as well as that of the $lx1$ vectors b^o and b^e, is made clear through the notation

D_r^o = a diagonal matrix of order l such that

$$(D_r^o)_i = (T_l^{-1})_i + \sum_{\nu=1}^{m_{odd}} d^{-1}_{i,2\nu-1}; \tag{66}$$

D_r^e = a diagonal matrix of order l such that

$$(D_r^e)_i = (T_l^{-1})_i + \sum_{\nu=1}^{m_{even}} d^{-1}_{i,2\nu}; \tag{67}$$

b^o = an $lx1$ vector such that

$$(b^o)_i = \sum_{\nu=1}^{m_{odd}} d^{-1}_{i,2\nu-1}\cdot(\text{RHS})_{i,2\nu-1} \quad \text{with } i = 1, 2, \ldots, l; \tag{68}$$

b^e = an $lx1$ vector such that

$$(b^e)_i = \sum_{\nu=1}^{m_{even}} d^{-1}_{i,2\nu}\cdot(\text{RHS})_{i,2\nu} \quad \text{with } i = 1, 2, \ldots, l. \tag{69}$$

Again it is noted that every (diagonal) element of D_r^o is greater than the sum of the elements of A'_{oo} and A'_{oe} lying on the same row and every (diagonal) element of D_r^e is greater than the sum of the elements of A'_{eo} and A'_{ee} lying on the same row with it.

All the above remarks lead to the conclusion that the linear system relating U^o, U^e, V^o, and V^e and composed of (62), (63), (64), and (65) is symmetric and—due to Gershgorin's theorem—positive definite. Next, V^o and V^e are expressed from (64) and (65) in terms of U^o and U^e and these expressions are inserted in (62) and (63). The final product is the symmetric system

$$[D_c^o - A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oo} - A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{eo}]U^o - [A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oe} + A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{ee}]U^e = a^o - A_{oo}\cdot(D_r^o)^{-1}\cdot b^o - A_{eo}\cdot(D_r^e)^{-1}\cdot b^e \tag{70}$$

$$-[A_{oe}\cdot(D_r^o)^{-1}\cdot A'_{oo} + A_{ee}\cdot(D_r^e)^{-1}\cdot A'_{eo}]U^o + [D_c^e - A_{oe}\cdot(D_r^o)^{-1}\cdot A'_{oe} - A_{ee}\cdot(D_r^e)^{-1}\cdot A'_{ee}]U^e = a^e - A_{oe}\cdot(D_r^o)^{-1}\cdot b^o - A_{ee}\cdot(D_r^e)^{-1}\cdot b^e. \tag{71}$$

However, due to the structure of the matrices A_{oo}, A_{eo}, A_{oe}, and A_{ee} as described by (54), (55), (56), and (57), respectively, both the products $A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oe}$ and $A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{ee}$ are equal to the null matrix of order m and consequently (70) and (71) become equivalent to the symmetric systems (72) and (73) of order m:

$$[D_c^o - A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oo} - A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{eo}]U^o = a^o - A_{oo}\cdot(D_r^o)^{-1}\cdot b^o - A_{eo}\cdot(D_r^e)^{-1}\cdot b^e \tag{72}$$

$$[D_c^e - A_{oe}\cdot(D_r^o)^{-1}\cdot A'_{oe} - A_{ee}\cdot(D_r^e)^{-1}\cdot A'_{ee}]U^e = a^e - A_{oe}\cdot(D_r^o)^{-1}\cdot b^o - A_{ee}\cdot(D_r^e)^{-1}\cdot b^e. \quad (73)$$

Both the linear systems (72) and (73), apart from being symmetric, are also positive definite. The proof for this, e.g., as far as the system (72) is concerned, might be sketched as follows:

Due to the structure of D_r^o and A_{oo} as expressed by relations (66) and (54), respectively, the sum $S_{2\kappa-1}$ of all the elements in the $2\kappa - 1$ row of the mxm matrix $A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oo}$—which is a matrix that contributes only to the odd-numbered rows of $[D_c^o - A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oo} - A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{eo}]$—i.e, the sum

$$S_{2\kappa-1} = \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa-1}^{-1}\cdot\left\{\sum_{\lambda=1}^{m_{\text{odd}}}[(D_r^o)^{-1}\cdot A'_{oo}]_{2\nu-1,2\lambda-1}\right\} \quad (74)$$

may be written in the form ($k = 1, 2, \ldots, m_{\text{odd}}$)

$$\begin{aligned} S_{2\kappa-1} &= \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa-1}^{-1}\cdot\sum_{\lambda=1}^{m_{\text{odd}}}\frac{d_{2\nu-1,2\lambda-1}^{-1}}{(T_l^{-1})_{2\nu-1}+\sum_{\rho=1}^{m_{\text{odd}}} d_{2\nu-1,2\rho-1}^{-1}} \\ &= \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa-1}^{-1}\cdot\left[1-\frac{(T_l^{-1})_{2\nu-1}}{(T_l^{-1})_{2\nu-1}+\sum_{\rho=1}^{m_{\text{odd}}} d_{2\nu-1,2\rho-1}^{-1}}\right] \\ &< \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa-1}^{-1}. \end{aligned} \quad (75)$$

On the other hand, using similar arguments one can prove that the sum $S_{2\kappa}$ of all the elements in the 2κ row of the mxm matrix $A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{eo}$—which is a matrix that contributes only to the even-numbered rows of the matrix $[D_c^o - A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oo} - A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{eo}]$—i.e., the sum

$$S_{2\kappa} = \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa}^{-1}\cdot\left\{\sum_{\lambda=1}^{m_{\text{even}}}[(D_r^e)^{-1}\cdot A'_{eo}]_{2\nu-1,2\lambda}\right\} \quad (76)$$

may be written as ($\kappa = 1, 2, \ldots, m_{\text{even}}$):

$$\begin{aligned} S_{2\kappa} &= \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa}^{-1}\cdot\sum_{\lambda=1}^{m_{\text{even}}}\frac{d_{2\nu-1,2\lambda}^{-1}}{(T_l^{-1})_{2\nu-1}+\sum_{\rho=1}^{m_{\text{even}}} d_{2\nu-1,2\rho}^{-1}} \\ &= \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa}^{-1}\cdot\left[1-\frac{(T_l^{-1})_{2\nu-1}}{(T_l^{-1})_{2\nu-1}+\sum_{\rho=1}^{m_{\text{even}}} d_{2\nu-1,2\rho}^{-1}}\right] \\ &< \sum_{\nu=1}^{l_{\text{odd}}} d_{2\nu-1,2\kappa}^{-1}. \end{aligned} \quad (77)$$

Having in mind (52), (75), and (77), one may now conclude that all the Gerschgorin circles of the symmetric (and for that reason with real eigenvalues) mxm matrix $[D_c^o - A_{oo}\cdot(D_r^o)^{-1}\cdot A'_{oo} - A_{eo}\cdot(D_r^e)^{-1}\cdot A'_{eo}]$ lie in the right-half plane, which in effect means that this matrix is positive definite.

Once U^o and U^e have been determined as solutions of the systems (72) and (73), respectively, relations (64) and (65) may be used to determine V^o and V^e through the formulas (D_r^o and D_r^e are diagonal)

$$V^o = (D_r^o)^{-1}\cdot[b^o - A'_{oe}U^e] \quad (77)$$

$$V^e = (D_r^e)^{-1}\cdot[b^e - A'_{eo}U^o - A'_{ee}U^e]. \quad (78)$$

Obviously, instead of choosing m as the order of the two linear systems for determining U^o and U^e, one may slightly modify the analysis so that first V^o and V^e are determined by solving two symmetric and positive definite linear systems of order l and then U^o and U^e are computed by means of (62) and (63), respectively. Aiming at the lowest possible order for the two symmetric and positive definite systems which lead to either (U^o, U^e) or (V^o, V^e), it is obvious that one should employ the procedure related to the minimum of the indices l and m.

Having U^o, U^e, V^o, and V^e it is now possible to compute all the elements of the lxm matrix X by means of (50) and, finally, the elements of the matrix Φ itself, using (43).

5. NUMERICAL EXPERIMENTS ASSOCIATED WITH FOURTH-ORDER SCHEMES

In this section both the quasi-direct method and the SOR iterative procedure are employed to solve four biharmonic problems of the first type in the square $0 \le x, y \le 1$. These problems are

Problem 1:

$$\nabla^2\nabla^2\phi = 8 \quad \text{with } \phi = x(1-x)\cdot y(1-y). \quad (80)$$

Problem 2:

$$\nabla^2\nabla^2\phi = 0 \quad \text{with } \phi = x^2 - y^2 + xe^x\cos y. \quad (81)$$

Problem 3:

$$\nabla^2\nabla^2\phi = 8[3x^2(1-x)^2 + 3y^2(1-y)^2 + (6x^2-6x+1)(6y^2-6y+1)] \quad \text{with } \phi = x^2(1-x)^2\cdot y^2(1-y)^2. \quad (82)$$

Problem 4:

$$\nabla^2\nabla^2\phi = (2\pi)^4[4\cos(2\pi x)\cos(2\pi y) - \cos(2\pi x) - \cos(2\pi y)] \quad \text{with } \phi = [1-\cos(2\pi x)][1-\cos(2\pi y)]. \quad (83)$$

TABLE I

Problem:	1	2	3	4
$1/h$	ε $\varepsilon_{[10]}$	ε $\varepsilon_{[10]}$	ε $\varepsilon_{[10]}$	ε $\varepsilon_{[10]}$
5	.11(−7) —	.55(−5) .94(−6)	.57(−4) .10(−4)	.19(0) .17(−1)
8	.56(−7) —	.43(−5) .98(−6)	.84(−5) .46(−6)	.26(−1) .26(−2)
10	.41(−7) —	(*)$_{dp}$ —	.30(−5)$_{dp}$ —	.10(−1) —
16	.34(−7) —	(*)$_{dp}$ *	(*)$_{dp}$ *	.16(−2) .15(−3)
20	.17(−6) —	(*)$_{dp}$ —	(*)$_{dp}$ —	.65(−3) —
25	.16(−6) —	(*)$_{dp}$ —	(*)$_{dp}$ —	.27(−3) —
32	.12(−6) —	(*)$_{dp}$ —	(*)$_{dp}$ —	.10(−3) —

All numerical tests were run on an AMSTRAD 1512 XT personal computer using a FORTRAN compiler with a (single) precision accuracy lying between 10^{-6} and 10^{-7}. A series of discretization steps $h = \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}$, and $\frac{1}{32}$ (after which the influence of rounding on the computations could not be avoided, even in double precision) has been selected and the results concerning the employment of the quasi-direct method and the SOR iterative procedure are summarized in Tables I and I_a, respectively.

The results consist of the maximum absolute discrepancy ε between the known and computed solutions and, in addition (when the SOR is used), the optimum value ω_{opt} of the relaxation parameter, as well as the number N of iterations required, so that the maximum absolute deviation in two consecutive computational stages equals 10^{-6}.

The above optimum value ω_{opt} is determined by an experimental approach. More precisely the corresponding computer program is run for a series of 10 relaxation parameters in the

TABLE I_a

Problem:	1		2		3		4	
1/h	ε	ω_{opt} N	ε	ω_{opt} N	ε	ω_{opt} N	ε	ω_{opt} N
5	.11(−7)	1.1 12	.55(−5)	1.0 17	.57(−4)	1.1 10	.19(0)	1.0 17
8	.56(−7)	1.0 18	.48(−5)	1.0 18	.85(−5)	1.1 15	.26(−1)	1.0 21
10	.71(−7)	0.9 20	(*)$_{dp}$		.31(−5)$_{dp}$	1.0 17	.10(−1)	1.0 25
16	.47(−7)	0.9 22	(*)$_{dp}$		(*)$_{dp}$		.16(−2)	1.0 36
20	.22(−6)	0.9 24	(*)$_{dp}$		(*)$_{dp}$		.65(−3)	1.0 42
25	.19(−6)	0.9 28	(*)$_{dp}$		(*)$_{dp}$		.27(−3)	1.0 49
32	.16(−6)	0.9 32	(*)$_{dp}$		(*)$_{dp}$		.11(−3)	1.0 60

interval 1.6 to 0.7 in steps of 0.1 and the one associated with the minimum number of iterations is recorded. From the data cited in Tables I and I_a it may be deduced that the observed accuracy (at least for the problems considered in this paper) is of fourth order.

In order to check the reliability of the solution a set of corresponding ε-results from [10]—designated as $\varepsilon_{[10]}$—have been included in Table I. At this point one may observe that the structure of (42) is such that the minimum of the indices l and m must not be less than 4 and consequently it is not possible to use a discretization step h $(=1/(l+1)$ or $1/(m+1))$ equal to $\frac{1}{4}$ as in [10]. For this reason a set of fictitious "results," $\varepsilon_{\frac{1}{5}}$, has been produced from the results $\varepsilon_{\frac{1}{4}}$ in [10] associated with $h = \frac{1}{4}$, so that, finally, $\varepsilon_{\frac{1}{5}}$ is set approximately equal to $= \varepsilon_{\frac{1}{4}} \cdot (4/5)^4$ after taking into account the fourth-order accuracy reported there.

In producing the results in [10], a DEC-20 computer has been used with a single-precision accuracy between 10^{-7} and 10^{-8}—see [2]—and the symbol * in Table I denotes the influence of rounding errors on the single-precision computations for that particular computer. On the other hand, the symbol $(\ )_{dp}$ denotes the employment of double-precision accuracy in the AMSTRAD 1512 XT personal computer so that the fourth-order accuracy of the particular problem can be maintained.

Finally the symbol $(*)_{s,dp}$ in some entries of both Tables I and I_a means that the results are influenced from rounding errors, not only in single, but in double precision as well, provided of course that an AMSTRAD 1512 is used. From the data presented in Tables I and I_a it is seen that the results, apart from showing fourth-order accuracy, exhibit more or less the same general behavior as those cited in [10] and, without being unacceptable, they are roughly one order of magnitude less accurate than the latest ones. The results referring to Problem 1 seem to be influenced only by rounding errors. Probably this is due to the fact that the structure of this problem is such that the difference equations of the non-coupled (p,q) model are exactly satisfied by its solution—see [5]—and perhaps this is somehow reflected in the relatively small values of ε. Similar behavior was shown in the problem $\nabla^2\nabla^2\phi = 0$ with $\phi = x^3 - 3y^2 + 2xy$ whose non-coupled (p,q) model is again satisfied exactly by its solution [6]. However, the results for this problem have not been included in the tables.

As a final remark on the results cited in Tables I and I_a one may note that the processing times for the quasi-direct method range from about 2.5 s ($h = \frac{1}{5}$ to 200 s ($h = \frac{1}{32}$) and that these times are roughly doubled when the SOR method is employed.

6. SECOND-ORDER SCHEMES

In deriving schemes with second-order truncation errors the discrete equivalents of $\nabla^2 g = f$ and $\nabla^2\phi = g$ are again combined in a single model, but now the Laplacian is approximated through 5-point stencils of the type

$$\begin{matrix} & h_y/h_x & \\ h_x/h_y & -2(h_y/h_x + h_x/h_y) & h_x/h_y \\ & h_y/h_x. & \end{matrix}$$

The above stencils are related to a rectangular grid having lxm interior nodes with mesh sizes $h_x = a/(l+1)$ and $h_y = b/(m+1)$, where l and m stand for the number of equidistant subdivision points of the edges a and b, respectively. The possibility of two different steps, h_x and h_y, may be proved advantageous for some peculiar values of the ratio a/b.

The process of constructing the matrix equation in this case is identical in principle to that described in Section 2, provided of course that the values of $g = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2$ at the boundaries are expressed as second-order linear combinations of either known or unknown ϕ-values and normal ϕ-derivatives at the boundaries in question. The most important components of such approximations are derived by means of the (1, 2)-approach—see [5,6]—and actually are relations of the type

$$(\partial^2\phi/\partial y^2)_{i,o} = [-7\phi_{i,o} + 8\phi_{i,1} - \phi_{i,2} - 6h_y(\partial\phi/\partial y)_{i,o}]/(2h_y^2) + O(h_y^2) \quad (84)$$

on which—like (4a)—the construction of the whole discrete model is finally based.

The form of the final version of the transformed discrete model of (1) with a second-order truncation error is analogous to (42) and it will be simply put down without any attempt to repeat again the basic steps leading to its derivation. However, before such a presentation can be possible, a number of matrices must be introduced which will clarify its meaning. Some of these matrices play the same role as that encountered in fourth-order models and for that reason even their names will remain unchanged, although they are used in a different context. When this is the case, the number of the relation giving the previous definition of the symbol concerning the fourth-order schemes will be recorded as well. In this way the following additional notation is introduced ($\mu = h_y/h_x$):

$$\text{LAPL}[\ \ ] = \mu A_l(-2, 1)[\ \ ] + \mu^{-1}[\ \ ]A_m(-2, 1) \quad (85)$$

$$\begin{aligned} F_{\text{lapl}} = (h_x h_y)^2 F &- \text{LAPL}[\Phi_{lr} + \Phi_{ud}] \\ &- (h_x h_y)[SD_{x,lr} + SD_{y,ud}] \\ &+ \tfrac{1}{2}\mu^{-1}[7\Phi_{lr} + 6h_y \cdot FD_{y,lr}] \\ &+ \tfrac{1}{2}\mu[7\Phi_{ud} + 6h_x \cdot FD_{x,ud}] \end{aligned} \quad (86)$$

E_ν = a diagonal matrix of order ν with elements equal to $\frac{1}{2}[16(V_\nu)_{i,1} - 2(V_\nu)_{i,2}]$ for $i = 1(1)\nu$; see also (32) (87)

$E_\nu^{-1}S_\nu^{-1} = T_\nu = Z_\nu^{-1}$ = a diagonal matrix of order ν with positive elements for $(p, q) = (1, 2)$; see also (39) and (40) (88)

$$\text{DLAPL}[\ \ ] = \mu D_l(-2, 1)[\ \ ] + \mu^{-1}[\ \ ]D_m(-2,1) \quad (89)$$

$$S_l^{-1}V_lF_{|\mathrm{lap}|}V_mS_m^{-1} = \mathrm{RHS}; \quad \text{see also (41)} \tag{90}$$

$$c_{ij} = (T_l)_i\{\mu[D(-2,1)]_i + \mu^{-1}[D_m(-2,1)]_j\}^2(T_m)_j + (T_l)_i + (T_m)_j; \quad \text{see also (45)} \tag{91}$$

or

$$c_{ij} = d_{ij} + (T_l)_i + (T_m)_j \quad \text{with } d_{ij} = (T_l)_m\{\mu[D(-2,1)]_i + \mu^{-1}[D_m(-2,1)]_j\}^2(T_m)_j; \quad \text{see also (45a).} \tag{91a}$$

By means of the already introduced additional symbolism, the second-order discrete model for the first biharmonic problem of (1) in a rectangle axb is written

$$T_l \cdot \mathrm{DLAPL}^2\,[X] \cdot T_m + T_lXQ_m + Q_lXT_m = \mathrm{RHS}. \tag{92}$$

The elements of the $l x m$ matrix X can be determined from (92) either by means of a quasi-direct method quite similar to that introduced in Section 4 or by employing the SOR iterative procedure. After X has been determined, the $l x m$ matrix Φ is computed using (43).

7. NUMERICAL EXPERIMENTS ASSOCIATED WITH SECOND-ORDER SCHEMES

In Tables II and IIa results concerning the processing of second-order schemes and derived by means of the quasi-direct and SOR methods, respectively, have been recorded. The same test problems 1, 2, 3, and 4 and the same domain $0 \le x,y \le 1$, introduced in Section 5 for fourth-order schemes, are used in the present case as well.

The discretization steps ($h_x = h_y = h$) now equal $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{16}$, $\frac{1}{20}$, $\frac{1}{25}$, and $\frac{1}{32}$, while the form of the data in Tables II and $\mathrm{II_a}$ is analogous to that referring to Tables I and $\mathrm{I_a}$, respectively. More precisely, the results consist of the maximum absolute discrepancy ε among the known and computed solutions and, in addition (when SOR is used), of the optimum value ω_{opt} of the relaxation parameter, as well as the (experimentally determined) number N of iterations required, so that the terminating criterion equals 10^{-6}.

In order to check the reliability of the quasi-direct method a number of corresponding ε-results from [6, 10] and designated as $\varepsilon_{[6]}$ and $\varepsilon_{[10]}$, respectively, have been included in Table II. The necessary checks for the SOR method refer to both the value of ε and the number N of iterations. For this reason in Table $\mathrm{II_a}$ the ε and N results from Ref. [5, 6] have been included which are designated as ($\varepsilon_{[5]}$, $\varepsilon_{[6]}$) and ($N_{[5]}$, $N_{[6]}$), respectively.

All the symbols *, $(\varepsilon)_{\mathrm{dp}}$, and $(*)_{\mathrm{s,dp}}$ that may appear in Tables II and $\mathrm{II_a}$ retain their meaning as described in Section 5. Double precision has been used when the results in single precision were not in agreement with the ones reported in Refs. [5, 6] which have been derived by means of a mainframe computer (e.g., an IBM 370/158 in [6]) or when the results themselves were influenced by rounding errors. The processing times—referring to an AMSTRAD 1512 XT personal computer—range from 1.5 s ($h = \frac{1}{4}$) to 55 s ($h = \frac{1}{32}$) when the quasi-direct method is used, and these times are roughly doubled for the case of the SOR method.

Finally one may observe from the data cited in Tables II and $\mathrm{II_a}$ that the accuracy of the schemes is at least of second order as it should be [5, 7].

8. CONCLUDING REMARKS

The computational schemes used in this paper with either fourth- or second-order truncation errors lead to results with accuracy of the same order as can be seen from the results of the numerical experiments. When the structure of the discrete model of (1) is associated with fourth-order truncation errors the observed results, although revealing fourth-order accuracy as well, are roughly one order of magnitude less accurate than the ones reported in [10].

As has been stated already in the Introduction the excellent accuracy reported in [10] may originate from the very structure of the discrete equivalent of (1) through the use of a triplet of nodal parameters which leads to a much better substitute for the original problem than models with a single nodal parameter. However, due to the increased number of unknowns, memory requirements, as well as the complicated structure of the resulting difference equations, limit the applicability of the method, while in cases where iterative procedures were used, convergence was very slow and sometimes it was not achieved at all.

It seems that the above weak points of the numerical processing in Stephenson's method constitute the price one must pay for its highly accurate results and that as stated in [10] "it is necessary to develop suitable solution techniquess that take advantage of the block structure of the resulting system of equations." Such a goal has been partly achieved by the solution techniques developed in this paper by presenting a quasi-direct solution method and a compact iterative scheme for systems of difference equations with one nodal parameter. Certainly it would be nice to extend these techniques to the type of models described in [10]. A possible solution might be the expression of Hermite's models in [10] as a system of three matrix equations and then to employ the method presented in [9].

As a matter of fact, it has been possible to formulate a system of three matrix equations of the type in question. Unfortunately the constant matrices participating in this formulation do not possess a common system of eigenvectors and, consequently, the extended form of the "irritational method" [1] is of no use.

Before leaving the subject of computational schemes with fourth-order truncation errors it must be noted that the author was not able to extend the scope of relation (4.5) in [7] or relation (15) in [11] which could lead to a theoretical determination of the accuracy of the (p, q, r, s) model as in the case of the (p, q) approach—see, e.g., [7]. It was precisely due to this

TABLE II

Problem:	1	2	3	4
1/h	ε	ε	ε	ε
	$\varepsilon_{[6]}$	$\varepsilon_{[6]}$	$\varepsilon_{[6]}$	$\varepsilon_{[6]}$
	$\varepsilon_{[10]}$	$\varepsilon_{[10]}$	$\varepsilon_{[10]}$	$\varepsilon_{[10]}$
4	.19(−7)	.14(−2)	.80(−3)	.14(+1)
	—	—	—	—
	—	.29(−4)	.39(−3)	.44(0)
5	.37(−8)	.70(−3)	.40(−3)	.65(0)
	—	.70(−3)	—	—
	—	—	—	—
8	.82(−7)	.19(−3)	.15(−3)	.34(0)
	—	—	—	—
	—	.76(−5)	.94(−4)	.10(0)
10	.30(−7)	.85(−4)	.87(−4)	.23(0)
	—	.88(−4)	—	—
	—	—	—	—
16	.22(−7)	.13(−4)	.30(−4)	.93(−1)
	—	.12(−4)	—	—
	—	*	.30(−5)	.26(−1)
20	.13(−6)	.49(−5)	.18(−4)	.58(−1)
	—	.47(−5)	.18(−4)	.60(−1)
	—	—	—	—
25	.20(−6)	.31(−5)$_{dp}$	.11(−4)	.36(−1)
	—	—	—	.39(−1)
	—	—	—	—
32	.16(−6)	.19(−5)	.63(−5)	.22(−1)
	—	—	—	—
	—	—	—	—

absence of a formal proof of the accuracy for schemes with fourth-order truncation errors that all the statements concerning the accuracy of models with 9-point stencils were made with the fourth-order accuracy results of Tables I and I$_a$ in mind and that not any attempt for more general conclusions followed.

As far as the computational schemes with second-order truncation errors one may observe that the results cited in Tables II and II$_a$ confirm the second-order accuracy proved in [7] and that when SOR is used the number of iterations needed is substantially less than the corresponding numbers reported in [5].

APPENDIX

In deriving a fourth-order linear combination of ϕ-values at the nodes (i, o), (i, p), (i, q), (i, r), and (i, s) to approximate e.g., $(\partial^2\phi/\partial y^2)_{i,o}$, the sum

$$S = \beta_o\phi_{i,o} + \beta_p\phi_{i,p} + \beta_q\phi_{i,q} + \beta r\phi_{i,r} + \beta_s\phi_{i,s} \quad \text{(A.1)}$$

is considered and each of the quantities $\phi_{i,p}$, $\phi_{i,q}$, $\phi_{i,r}$, and $\phi_{i,s}$ is replaced by its Taylor series expansion about the boundary node (i, o). For the expansion in question, which includes derivatives at (i, o) up to the fifth order, one seeks to determine such values of β_p, β_q, β_r, and β_s so that the coefficients multiplying third, fourth, and fifth derivatives about (i, o) will equal zero.

The latest requirement leads finally to the system

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & q^2 & r^2 \end{bmatrix} \begin{bmatrix} p^3\beta_p \\ q^3\beta_q \\ r^3\beta_r \end{bmatrix} = -s^3\beta_s \cdot \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} \quad \text{(A.2)}$$

whose coefficient matrix is a Vandermonde matrix of order 3. Inversion of the latest system, after the basic properties of the Vandermonde matrices have been taken into account, results finally in the solution

$$\begin{bmatrix} p^3\beta_p \\ q^3\beta_q \\ r^3\beta_r \end{bmatrix} = -\{s^3\beta_s/(r-q)(r-p)(q-p)\}$$

TABLE II$_a$

Problem:	1		2		3		4	
$1/h$	ε $\varepsilon_{[5]}$ $\varepsilon_{[6]}$	ω_{opt} N $N_{[5]}$ $N_{[6]}$	ε $\varepsilon_{[5]}$ $\varepsilon_{[6]}$	ω_{opt} N $N_{[5]}$ $N_{[6]}$	ε $\varepsilon_{[5]}$ $\varepsilon_{[6]}$	ω_{opt} N $N_{[5]}$ $N_{[6]}$	ε $\varepsilon_{[5]}$ $\varepsilon_{[6]}$	ω_{opt} N $N_{[5]}$ $N_{[6]}$
4	.19(−7)	1.0	.14(−2)	1.1	.80(−3)	1.0	.14(+1)	1.0
		7		9		6		9
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
5	.29(−7)	1.0	.70(−3)	1.1	.40(−3)	1.1	.65(0)	1.1
		9		9		8		12
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
8	.22(−7)	1.0	.19(−3)	1.1	.15(−3)	1.0	.34(0)	1.0
		12		13		11		17
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
10	.40(−7)	1.0	.87(−4)	1.1	.87(−4)	1.0	.23(0)	1.1
		13		15		11		18
	—	23	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
16	.90(−7)	1.0	.16(−4)	1.0	.31(−4)	1.1	.90(−1)	1.0
		15		23		13		24
	—	—	.23(−4)	42	—	—	—	—
	—	—	.23(−4)	42	—	—	—	—
20	.83(−6)	1.0	.52(−5)	1.0	.19(−4)	1.0	.58(−1)	1.0
		16		25		15		28
	—	31	—	—	—	—	—	—
	—	—	.29(−4)	48	—	—	—	—
25	.89(−6)	1.0	.32(−5)$_{dp}$	1.0	.11(−4)	1.0	.36(−1)	1.0
		18		28		16		33
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
32	.95(−6)	1.0	.20(−5)$_{dp}$	1.0	.65(−5)	1.0	.22(−1)	1.0
		21		33		18		38
	—	—	.26(−5)	63	—	—	—	—
	—	—	—	—	—	—	—	—

$$\cdot \begin{bmatrix} r-q & 0 & 0 \\ 0 & r-p & 0 \\ 0 & 0 & q-p \end{bmatrix} \begin{bmatrix} rq & -(r+q) & 1 \\ -rp & (r+p) & -1 \\ qp & -(q+p) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} \quad \text{(A.3)}$$

from which relations

$$\beta_p = -\beta_s (s/p)^3 [(s-r)(s-q)/(r-p)(q-p)]$$
$$\beta_q = +\beta_s (s/q)^3 [(s-r)(s-p)/(r-q)(q-p)] \quad \text{(A.4)}$$
$$\beta_r = -\beta_s (s/r)^3 [(s-q)(s-p)/(r-q)(r-p)]$$

emerge. Since p, q, r, and s are integers, one sets $\beta_s = -(rpq)^3 (r-q)(r-p)(q-p)$ so that β_p, β_q, and β_r will take integer values as well. In this way,

$$\begin{aligned} \beta_p &= +(srq)^3(s-r)(s-q)(r-q) \\ \beta_q &= -(srp)^3(s-r)(s-p)(r-p) \\ \beta_r &= +(sqp)^3(s-q)(s-p)(q-p) \\ \beta_s &= -(rqp)^3(r-q)(r-p)(q-p) \end{aligned} \quad \text{(A.5)}$$

and the final fourth-order formula for $(\partial^2\phi/\partial y^2)_{i,0}$ takes the form

$$(\partial^2\phi/\partial y^2)_{i,o} = \{-(\beta_p + \beta_q + \beta_r + \beta_s)\phi_{i,o} + \beta_p\phi_{i,p} + \beta_q\phi_{i,q} + \beta_r\phi_{i,r} + \beta_s\phi_{i,s} - h(p\beta_p + q\beta_q + r\beta_r + s\beta_s)(\partial\phi/\partial y)_{i,o}\}/ [0.5\,h^2(p^2\beta_p + q^2\beta_q + r^2\beta_r + s^2\beta_s)].$$

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